Determining the size of jurisdictions for implementing language rights

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REAL

Research group “Economics and language”

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Abstract

Language policy is modeled as a collection of planning measures with different properties with respect to their importance for the beneficiaries and the implementation costs. The costs depend on the number of beneficiaries of the policy as well as on the area of implementation. The geographical distribution of the beneficiaries as well as the size of the area characterize the potential jurisdiction where the policy would be implemented.

It is analyzed, how the size of the jurisdictions should be chosen in order to guarantee beneficial minority rights. It is found, among other things, that jurisdictions should be limited in size if planning measures are both spatial and rival; that jurisdictions for important measures should be larger than for less important ones; and that more extensive rights are possible for geographically concentrated minorities than for uniformly distributed ones.

Keywords: Constitutional economics, language rights, language policy, policy analysis, costs

JEL classification:

1 INTRODUCTION

We define language policy as a collection of different language-planning measures. Such a measure could be the publication of official documents in a minority language, bilingual street signs, the elementary school system being offered in a minority language etc. These measures show very different cost characteristics. The publication of official documents in Internet in a given language cause mainly fixed costs that are independent of the number of beneficiaries as well as the spatial size of the jurisdiction concerned. The costs of providing elementary education in a minority language, on the other hand, strongly depend on both the number of beneficiaries and the physical size of the jurisdiction, where the schools are located. The costs of bilingual street signs do not depend on the number of people orienting themselves with the help of those signs, but the size of the territory strongly influences the costs.

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In determining if a policy is efficient or not, one has to compare its benefits with the costs, possibly taking distributional effects into account. A fair approximation of the benefits is assuming them proportional to the number of beneficiaries. Because of the different cost structures and different size of the *per capita* benefits of different measures, there will be no simple planning rule for the implementation of the various measures fitting them all. In WICKSTRÖM (2017) we analyzed how to group different measures into categories and how to find simple policy rules for the implementation of the different categories. Basically, the structure of the costs as well as the size of the benefits *per capita* are the crucial parameters for organizing measures into categories for a given jurisdiction.

In this essay, we turn the problem around and look at the implications of the cost structures and importance of the planning measures for the size of the jurisdictions. We want to caracterize a sensible administrative division of a state into different jurisdictions with their own different planning rules.

We first present the basic structure, which is very similar to that of WICKSTRÖM (2017). Then this structure is applied to the analysis of the desirable size of the jurisdiction in which the planning measure is implemented. The demographic characteristics of the state thereby play an important rôle.

2 BASIC MODEL

We first model the value of language-planning measures in a given jurisdiction and then the structure of the jurisdiction.

2.1 LANGUAGE-PLANNING MEASURES

Let a be the size of a jurisdiction and n the size of its minority population. The implementation costs of a language-planning measure is written as a concave function $c(n, a)$. The *per capita* benefits of the measure are given by β; that is, the gross benefits are βn. The net benefits are then:

$$u = \beta n - c(n, a) \tag{2.1}$$

A planning measure increases efficiency in society if u is positive. If this is the case, of course, defends on how the costs depend on n and a. If the costs do not depend on n, we call the measure non-rival, and if they do not depend on a, the measure is non-spatial. In general, a measure can display different degrees of rivalry and spatial dependence. An example of a spatial and rival measure is public schools in a minority language; Publishing official documents in Internet in a minority language is a good example of a non-spatial and non-rival measure; bilingual street signs are non-rival and spatial; and the introduction of a call center is a non-spatial and rival measure. The efficiency of measures with different cost structures will clearly depend on the demographics and size of the jurisdiction analyzed.

It turns out that it is convenient to make a change of variables, replacing a with the density of the minority population in the jurisdiction, δ:

$$\delta := \frac{n}{a} \tag{2.2}$$

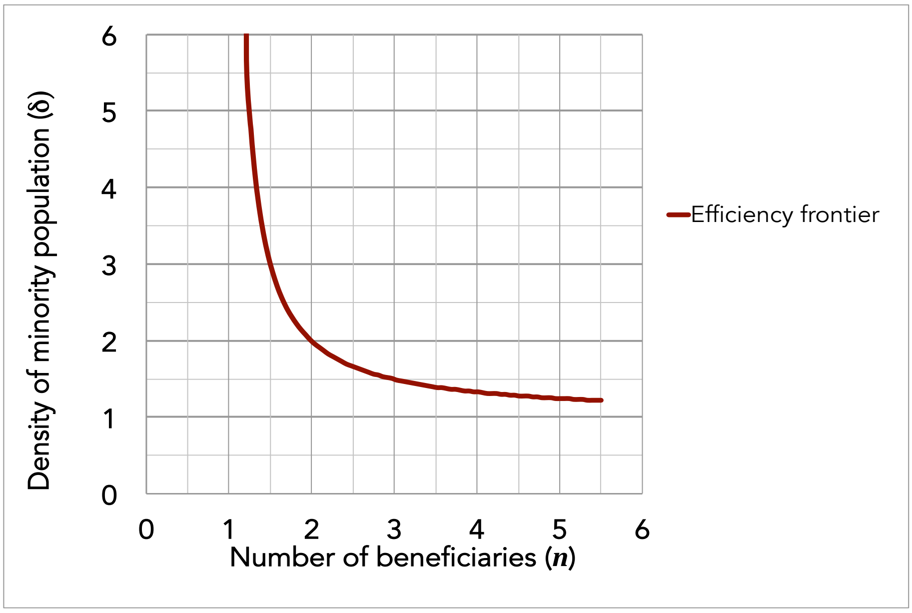

FIGURE 2.1 *An efficiency frontier of a rival and spatial measure with fixed costs*

This gives us:

$$u = \beta n - c\left(n, \frac{n}{\delta}\right) := \beta n - \tilde{c}(n, \delta) \tag{2.3}$$

We divide by β and define:

$$g(n, \delta) = n - \frac{\tilde{c}(n, \delta)}{\beta} \tag{2.4}$$

The measure under consideration is efficient if $g \geq 0$ and inefficient if $g < 0$. The border between efficiency and inefficiency, the efficiency frontier, $\delta^E(n)$, is then implicitly given by the equation:

$$g(n, \delta^E(n)) = 0 \tag{2.5}$$

In WICKSTRÖM (2017) it is shown that the language-policy analysis can be reduced to the two-dimension problem analzying the efficiency frontiers of the different planning measures in combination with the importance of the measures captured in the β in the jurisdiction under consideration.[1] The efficiency frontier turns out to be non-increasing in the $(\delta - n)$-diagram and divides the possible jurisdiction characterized by the size of their minority population and its density into those for which the the planning measure is efficient (to the North-East of the efficiency frontier) and those for which it is inefficient (to the South-West). In figure 2.1 a typical efficiency frontier is given. In this illustration the costs of the language-planning measure display rivalry and spatial dependence. In addition, there are fixed costs in the provision of the measure.

[1] See also WICKSTRÖM, TEMPLIN, and GAZZOLA (2018) and WICKSTRÖM, GAZZOLA, and TEMPLIN (2018).

2.2 THE SIZE THE JURISDICTION

For the sake of simplicity, we let the jurisdiction under consideration have unity width and variable length $a \in [0, a_M]$, where a_M is the size of the country. That is, the area of the jurisdiction is also a. The local density of the minority population is given by the (differentiable) function $\Delta(a)$. We assume that the minority is living in the area defined by $a \in [0, a_0]$ with $a_0 \leq a_M$ and that its density is non-increasing in a:[2]

$$\begin{aligned} \Delta(a) &> 0 \quad \text{for } 0 \leq a < a_0 \\ \Delta(a) &= 0 \quad \text{for } a_0 \leq a \leq a_M \\ \frac{\partial \Delta}{\partial a} &\leq 0 \quad \text{for } 0 \leq a \leq a_M \end{aligned} \tag{2.6}$$

The size of the minority population in the jurisdiction is denoted by the function $n(a)$:

$$n(a) = \int_0^a \Delta(x)\mathrm{d}x \tag{2.7}$$

The total size of the minority population, N, in the country under consideration is given by:

$$N = n(a_M) = n(a_0) \tag{2.8}$$

It is readily seen that $n(a)$ is concave on the interval $[0, a_M]$.

The average density of the minority population (δ) in the jurisdiction can be written as a function of the size of the jurisdiction a:

$$\delta = \tilde{\delta}^J(a) = \frac{n(a)}{a} = \frac{1}{a}\int_0^a \Delta(x)\mathrm{d}x \tag{2.9}$$

The function $\tilde{\delta}(a)$ is decreasing due to the concavity of $n(a)$.

The efficient policy, however, depends on the size of the parameters n and δ. In equations 2.7 and 2.9, n and δ are parametrized by a. Since $n(a)$ is strictly increasing on the interval $[0, a_0)$, the reverse function $n^{-1}(n)$ exists on this interval and is also strictly increasing. We can then define the function $\delta^J(n)$ and the corresponding a on the interval $[0, N)$:

$$\begin{aligned} \delta^J(n) :&= \frac{n}{n^{-1}(n)} \quad \text{for } 0 \leq n < N \\ a(n) &= n^{-1}(n) \quad \text{for } 0 \leq n < N \end{aligned} \tag{2.10}$$

The fact that $\tilde{\delta}(a)$ is decreasing implies that $\delta(n)$ is also decreasing on the interval $[0, N)$. In appendix A it is shown that $\delta^J(n)$ is concave if $\Delta(a)$ is concave.

In figure 2.2, the efficiency frontier of a planning measure and the location of a jurisdiction in the $(\delta - n)$-diagram in dependence of the size of the minority population n are illustrated. Also the fraction of the state territory that is taken up by the jurisdiction for different values of n is indicated.

[2] Imagine the jurisdiction to begin in the South on a border river and extend towards the North. The closer is the river, the higher is the concentration of the minority. Think of the US-Mexican border or the Slovak-Hungarian one.

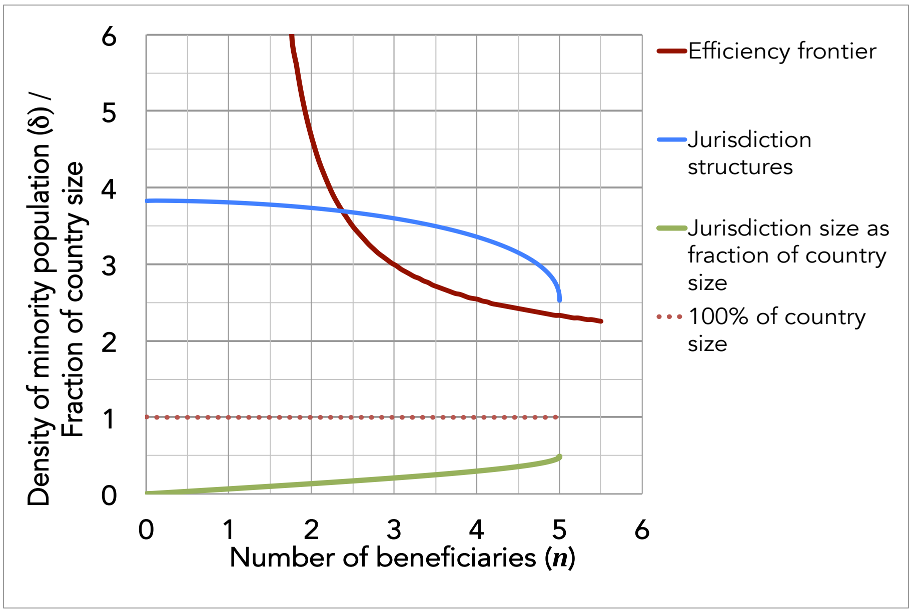

FIGURE 2.2 Efficiency frontier and minority structure of different sized jurisdictions

For the sake of illustration, we have specified $\Delta(a)$ as:

$$\Delta(a) = \Delta_0 - ba^{\varepsilon} \tag{2.11}$$

with $\varepsilon \in [0, \infty]$. Then:

$$n(a) = \int_0^a \Delta(x)\mathrm{d}x = \Delta_0 a - b\frac{1}{1+\varepsilon}a^{1+\varepsilon} \tag{2.12}$$

and

$$\tilde{\delta}^J(a) = \frac{n(a)}{a} = \Delta_0 - \frac{b}{1+\varepsilon}a^{\varepsilon} \tag{2.13}$$

In figure 2.2, $N = 5$, $\varepsilon = 2$ and $b = 1$. Δ_0 is determined residually and equal to 3.83. The value of the planning measure is increasing in the difference of the curves giving the minority structure of the jurisdiction and the efficiency frontier. In fact, it is given as a function of n by $\beta g(n, \delta^J(n))$. It is clear that the policy is beneficial only for for the values of n for which $\delta^J(n) \geq \delta^E(n)$. This in turn implies different values of a.

To find an exact expression for the optimal size of a jurisdiction we let a increase by a small amount, $\mathrm{d}a$. This will increase the minority population in the jurisdiction by:

$$\mathrm{d}n(a) = \Delta(a)\mathrm{d}a \geq 0 \tag{2.14}$$

We are interested in the optimal size, $\hat{a}$, of the jurisdiction. It is sensible to move the border if thereby the benefits of the language rights in effect increase more than the costs of implementing those rights. The change in gross benefits (b) of a policy measure due to a small change in the border, da, are given by:

$$\mathrm{d}b(a) = \beta \mathrm{d}n = \beta \Delta(a) \mathrm{d}a \qquad (2.15)$$

with β the average *per capita* benefit of the of the policy measure among the members of the minority. The increase in costs due to the change in the border location is:

$$\mathrm{d}c(a) = \frac{\partial c}{\partial a}\mathrm{d}a + \frac{\partial c}{\partial n}\mathrm{d}n = \frac{\partial c}{\partial a}\mathrm{d}a + \frac{\partial c}{\partial n}\Delta(a)\mathrm{d}a \qquad (2.16)$$

The change in net benefits becomes:

$$\mathrm{d}u(a) = \left[\beta\Delta(a) - \frac{\partial c}{\partial a} - \frac{\partial c}{\partial n}\Delta(a)\right]\mathrm{d}a = \left[\left(\beta - \frac{\partial c}{\partial n}\right)\Delta(a) - \frac{\partial c}{\partial a}\right]\mathrm{d}a \qquad (2.17)$$

The change is non-negative if:

$$\left(\beta - \frac{\partial c}{\partial n}\right)\Delta(a) \geq \frac{\partial c}{\partial a} \qquad (2.18)$$

This is our decision criterion for finding an optimal size of the jurisdiction.

3 OPTIMAL SIZE OF A JURISDICTION

Using 2.18 and the assumption that $\Delta(a)$ decreases in a we can find the optimal size of a jurisdiction for a policy measure, $\hat{a}$, provided that the measure is at all sensible. Formally, we write:

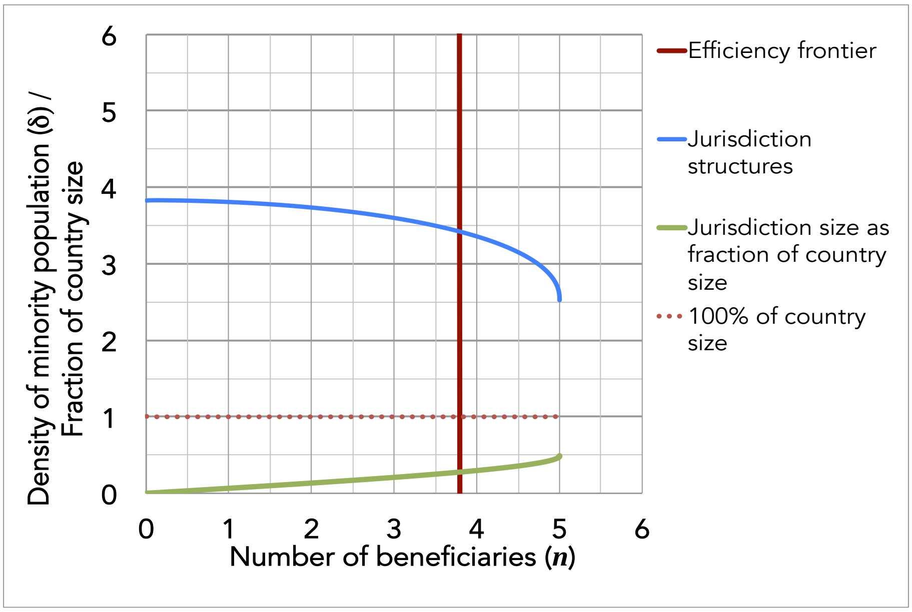

FIGURE 3.1 Non-spatial planning measure that will be implemented in the entire area with a minority population

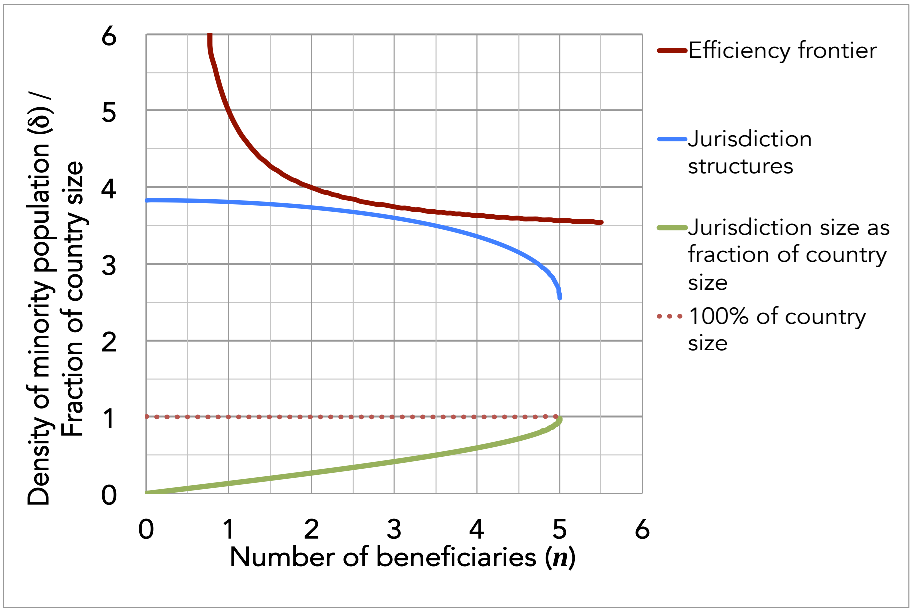

FIGURE 3.2 A spatial and rivaling planning measure that will not be implemented in a jurisdiction of any size

PROPOSITION 3.1 Let a^* be the largest feasible a satisfying expression 2.18. Then a^* maximizes the net benefit of the measure under consideration, $u(a)$. If $u(a^*) \geq 0$, $\hat{a} = a^*$ and is the optimal size of the jurisdiction for the measure. If $u(a^*) < 0$, the optimal size is $\hat{a} = 0$; that is, the measure should not be implemented.

The optimal size depends on the local density of the minority population as well as on the cost structure and the importance of the measure.

3.1 DEPENDENCE OF THE OPTIMAL SIZE OF THE JURISDICTION ON THE COST STRUCTURE OF THE POLICY MEASURE

We assume that policy measures bring the same gross benefits *per capita*, β, but differ in cost structures. We also assume that the population structure is the same for different measures. The cost structure is captured in expression 2.18 by the dependence of costs on the area of application, $\frac{\partial c}{\partial a}$, and on the dependence on the number of beneficiaries, $\frac{\partial c}{\partial n}$, If the costs of the policy measure is independent of a, expression 2.18 reduces to:

$$\left(\beta - \frac{\partial c}{\partial n}\right) \geq 0 \tag{3.1}$$

We note that concavity of the cost function implies that $\frac{\partial^2 c}{\partial n^2} \leq 0$ and conclude that only a “bang-bang” solution is possible:

PROPOSITION 3.2 If the policy measure is non-spatial, $\frac{\partial c}{\partial a} = 0$, the density of the minority population does not directly influence the size of the optimal jurisdiction.

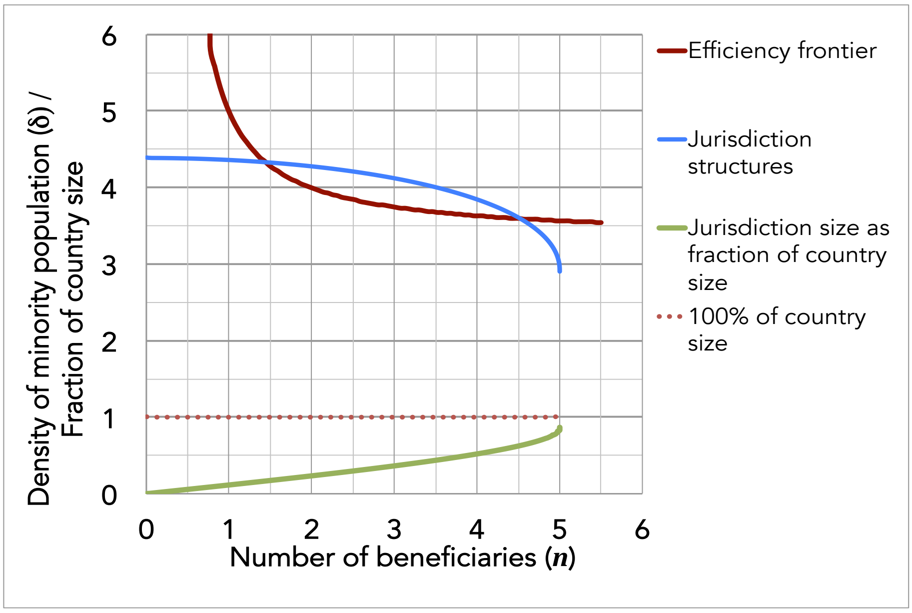

FIGURE 3.3 A spatial and rivaling planning measure that will be implemented in jurisdictions of some sizes

The optimal size of the jurisdiction is either the total area with a minority population, $\hat{a} = a_0$, or the measure should not be implemented, $\hat{a} = 0$.

This is illustrated in figure 3.1 for the optimal size being a_0, the entire area with a minority population which in the figure is half the country.

If the costs structure is spatial, the situation is more complex. We note that as the size of the jurisdiction, a, increases, so does n. *Ex hypothesi* the local density Δ then decreases, and due to the concavity of the cost function both derivatives in expression 2.18 also decrease. We can not determine the behavior of the left-hand side of the expression, since the expression in the parenthesis tends to increase and Δ tends to decrease. However, we know that if $a_0 < a_M$, for $a = a_0$ the left-hand side is zero and, hence, $a^* < a_0$. It might even be zero.

PROPOSITION 3.3 If $a_0 < a_M$, there exists an a^* less than a_0 such that the optimal size of the jurisdiction $\hat{a} = a^*$ or the measure should not be implemented, $\hat{a} = 0$. The more rival or spatial is the measure, *ceteris paribus*, the smaller is the optimal size of the jurisdiction.

This is illustrated for the case of the optimal size being equal to zero in figure 3.2 and for a positive optimal size when the minority is more concentrated in figure 3.3. If the concentration of the minority population is not to strong, the bang-bang result could reemerge; see figure 3.4:

PROPOSITION 3.4 If $a_0 = a_M$ and the minority population is relatively evenly distributed over the whole country, a^* could equal a_M and $\hat{a}$ is either equal to a_M or zero.

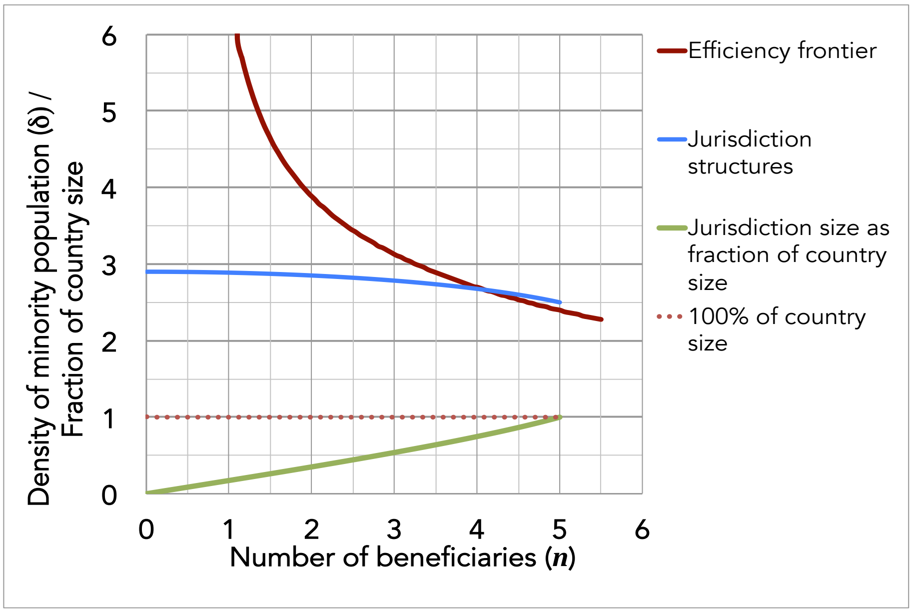

FIGURE 3.4 A spatial and rivaling planning measure that will be implemented in a jurisdiction spanning the entire country

3.2 THE IMPORTANCE OF A MEASURE AND THE OPTIMAL SIZE OF JURISDICTIONS

The importance of a policy measure is reflected in the *per capita* gross benefit β. It is then obvious that a^* increases with the importance of the policy measure:

> PROPOSITION 3.5 In the case of the optimal size of the jurisdiction lying between zero and a_M, an increase in the importance of the measure will cause the optimal size of the jurisdiction to increase.

In figure 3.5 the importance of the measure in figure 3.2 has increased.

3.3 THE POPULATION STRUCTURE AND THE OPTIMAL SIZE OF JURISDICTIONS

Maybe the most interesting case is that due to changes in the population structure. The migration of members of the minority away from traditional areas into cities and towns away from the homelands does not change the size of the minority population, but the spacial structure. The question is to find the implication for the optimal jurisdiction of this change in the densities of the minority.

It is clear that the local density $\Delta(a)$ will decrease for small a, say if $a < \tilde{a}$, and increase if $a > \tilde{a}$. Since the integral 2.7 has to have the same value for $a = a_0$ in all cases, namely N, $n(a)$ will decrease for all $a < a_0$ and a_0 might increase. That is, the inverse function $n^{-1}(n)$ will increase for all $n < N$. This implies that $\delta(n)$ will decrease for all $n < N$. The sizes of the jurisdiction making the planning measure efficient will be more restrictive. Indeed, many rival and spatial measures will have efficiency frontiers above the curve describing the possible jurisdictions, making them inefficient for any size of the jurisdiction. Only the non-rival measures will be unaffected by the change in the population structure, since they need a critical

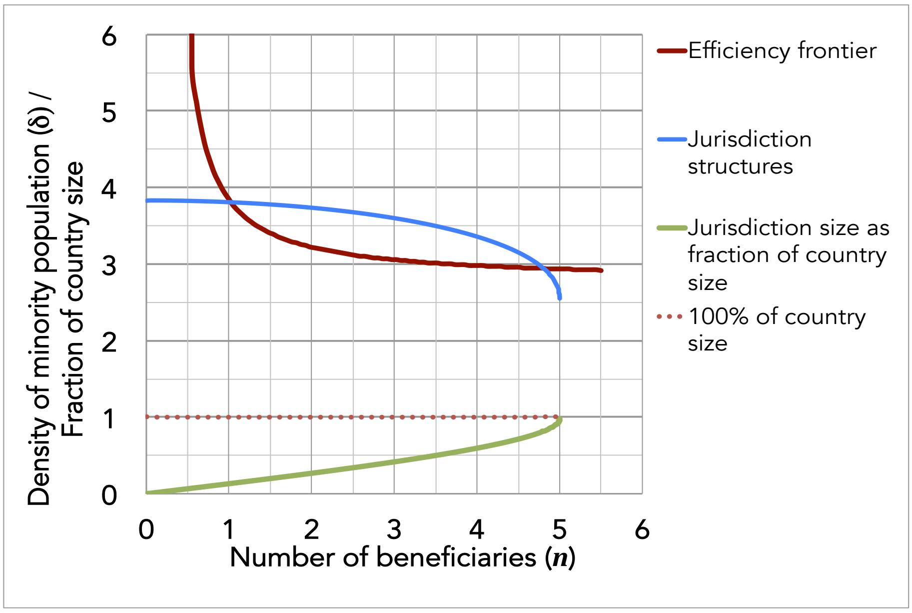

FIGURE 3.5 Important spatial and rivaling planning measure that will be implemented in jurisdictions of some sizes

number of beneficiaries in order to be efficient and the total size of the minority remains at N, making any measure with a critical number less than or equal to N efficient in the jurisdiction consisting of the entire country.

PROPOSITION 3.6 A reduction in the concentration of a minority will lead to fewer language planning measure being efficient and hence to less language rights for the minority. Only non-spatial measures are unaffected.

Figure 3.6 illustrates the case of a homogeneously distributed minority with the planning measure of figure 3.5.

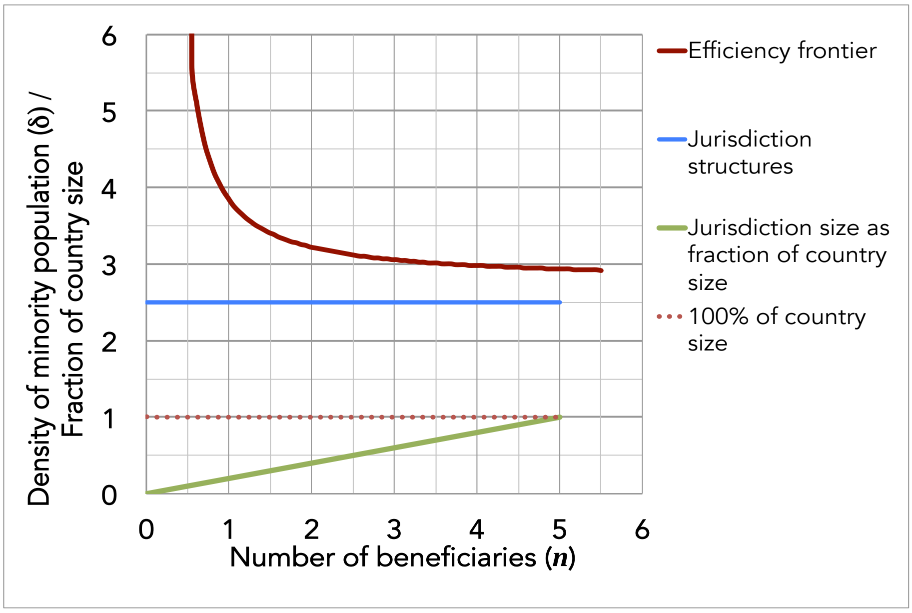

FIGURE 3.6 A spatial and rivaling planning measure that is not implemented for a uniformly distributed minority

4 CONCLUDING REMARKS

The discussion above concerned only one planning measure. For a comprehensive study, we would have to look at categories of measures and how the change in the jurisdiction structure affect the choice of the optimal composition of categories. That is, which measures should be collected in each category and how many categories should be defined.

What is certain, is that more flexibility, both in the territorial structure and in the construction of language policy and planning – not just having one category "official language", but several depending on the cost structures and benefits of the single measures – can improve the situation for everyone.

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APPENDIX

A PROOF OF CONCAVITY OF THE LOCI OF POSSIBLE JURISDICTIONS IN THE $(\delta - n)$-DIAGRAM

We normalize the $\Delta(a)$ function, such that $\Delta(a) = 0$ for $a \geq 1$ and $\Delta(a) > 0$ for $a < 1$ and such that:

$$\int_0^1 \Delta(x)\mathrm{d}x = N = 0.5 \tag{A.1}$$

Further, let $\Delta(a)$ be a concave function of a for $0 \leq a \leq 1$ and define:

$$n(a) := \int_0^a \Delta(x)\mathrm{d}x, \quad 0 \leq a \leq 1 \tag{A.2}$$

Let the inverse function be $n^{-1}(n(a)) = a$, for $0 \leq n \leq 0.5$, $\tilde{\delta}(a) := n(a)/a$, and $\delta(n) := \tilde{\delta}(n^{-1}(n)) = n/n^{-1}(n)$, for $0 \leq n \leq 0.5$. We want to show:

PROPOSITION A.1 $\delta(n)$ is a concave function on $0 \leq n \leq 0.5$

The proof is made in several steps. We first make some definitions:

DEFINITION A.1 For $0 \leq k < 1$:

$$\Delta_k(a) = \begin{cases} \frac{1}{k+1}, & 0 \leq a \leq k \\ \frac{1-a}{1-k^2}, & k \leq a \leq 1 \end{cases} \tag{A.3}$$

For $k = 1$:

$$\Delta_1(a) = \frac{1}{2} \tag{A.4}$$

Δ_k clearly satisfies the properties of the general Δ.

DEFINITION A.2 For $0 \leq k < 1$:

$$n_k(a) = \begin{cases} \frac{a}{k+1}, & 0 \leq a \leq k \\ \frac{a}{1-k^2} - \frac{1}{2}\frac{a^2+k^2}{1-k^2}, & k \leq a \leq 1 \end{cases} \tag{A.5}$$

For $k = 1$:

$$n_1(a) = \frac{1}{2} \tag{A.6}$$

DEFINITION A.3

$$n_k^{-1}(n) = \begin{cases} (k+1)n, & 0 \leq a \leq \frac{k}{1+k} \\ 1 - \sqrt{(1-k^2)(1-2n)}, & \frac{k}{1+k} \leq n \leq 0.5 \end{cases} \tag{A.7}$$

DEFINITION A.4 For $0 \le k < 1$:

$$\tilde{\delta}_k(a) = \begin{cases} \frac{1}{k+1}, & 0 \le a \le k \\ \frac{1-\frac{1}{2}a-\frac{1}{2}\frac{k^2}{a}}{1-k^2}, & k \le a \le 1 \end{cases} \tag{A.8}$$

For $k = 1$:

$$\tilde{\delta}_1(a) = \frac{1}{2} \tag{A.9}$$

$\delta_k(n)$ and n_k^{-1} are almost everywhere differentiable (except at $n = k/(1+k)$ and $n = 0.5$). We prove:

LEMMA A.1 δ_k is non-increasing and concave.

PROOF We calculate the derivative of $\delta_k(n)$:

$$\frac{\partial \delta_k}{\partial n} = \begin{cases} 0, & 0 \le n \le \frac{k}{1+k} \\ \frac{1}{2}\frac{(k/a)^2-1}{1-k^2}\frac{\partial a}{\partial n}, & \frac{k}{1+k} \le n < 0.5 \end{cases} \tag{A.10}$$

and of $n^{-1}(n)$:

$$\frac{\partial a}{\partial n} = \begin{cases} k+1, & 0 \le a \le \frac{k}{1+k} \\ \frac{1-k^2}{\sqrt{(1-k^2)(1-2n)}}, & \frac{k}{1+k} \le n < 0.5 \end{cases} \tag{A.11}$$

Since $0 \le k < 1$, $\frac{\partial a}{\partial n} \ge 0$ and as a consequence $\frac{\partial \delta_k}{\partial n} \le 0$.

To prove the concavity we find the second derivatives:

$$\frac{\partial^2 \delta_k}{\partial n^2} = \begin{cases} 0, & 0 \le n \le \frac{k}{1+k} \\ -\frac{k^2/a^3}{1-k^2}\left(\frac{\partial a}{\partial n}\right)^2 + \frac{1}{2}\frac{(k/a)^2-1}{1-k^2}\frac{\partial^2 a}{\partial n^2}, & \frac{k}{1+k} \le n < 0.5 \end{cases} \tag{A.12}$$

and:

$$\frac{\partial^2 a}{\partial n^2} = \begin{cases} 0, & 0 \le a \le \frac{k}{1+k} \\ \frac{(1-k^2)^2}{((1-k^2)(1-2n))^{1.5}}, & \frac{k}{1+k} \le n < 0.5 \end{cases} \tag{A.13}$$

It is readily seen that $\frac{\partial^2 a}{\partial n^2} \ge 0$. Hence, $\frac{\partial^2 \delta_k}{\partial n^2} \le 0$ and δ_k is concave. ■

Next, we show that any weighted sum of the $\Delta_k(a)$ also lead to a concave $\delta(n)$. We define:

DEFINITION A.5

$$\Delta^M(a) := \sum_{m=0}^{M} \gamma_m \Delta_{k_m}, \quad \gamma_m \ge 0, k_0 = 0, k_M = 1, k_{m+1} > k_m \tag{A.14}$$

It is clear that $\Delta^M(a)$ is non-increasing and concave. We can now prove:

LEMMA A.2 The $\delta^M(n)$ corresponding to $\Delta^M(a)$ is non-decreasing and concave.

PROOF We find the associated $n^M(a)$ and $\tilde{\delta}^M(a)$:

$$n^M(a) = \int_0^a \sum_{m=0}^{M} \gamma_m \Delta_{k_m}(x)\mathrm{d}x = \sum_{m=0}^{M} \gamma_m n_{k_m}(a) \tag{A.15}$$

$$\tilde{\delta}^M(a) = \frac{n(a)}{a} = \sum_{m=0}^{M} \gamma_m \frac{n_{k_m}}{a} = \sum_{m=0}^{M} \gamma_m \tilde{\delta}_{k_m}(a) \tag{A.16}$$

Since $\tilde{\delta}_{k_m}$ changes from a constant to a decreasing function at the point k_m, it is clear that the (negative) slope of $\tilde{\delta}^M$ decreases at each such point if $\tilde{\delta}_{k_m}$ has positive weight, $\gamma_m > 0$. To show the concavity of $\delta^M(n)$ we have to find its behavior between the points k_m. Between those points it is differentiable:

$$\frac{\partial \delta^M}{\partial n} = \sum_{m=0}^{M} \gamma_m \frac{\partial \tilde{\delta}_{k_m}}{\partial a} \frac{\partial a^M}{\partial n} \tag{A.17}$$

and

$$\frac{\partial^2 \delta^M}{\partial n^2} = \sum_{m=0}^{M} \gamma_m \left[\frac{\partial^2 \tilde{\delta}_{k_m}}{\partial a^2} \left(\frac{\partial a^M}{\partial n} \right)^2 + \frac{\partial \tilde{\delta}_{k_m}}{\partial a} \frac{\partial^2 a^M}{\partial n^2} \right] \tag{A.18}$$

We differentiate expression A.15, in order to find the derivative of $n^{M^{-1}}(n)$:

$$1 = \sum_{m=0}^{M} \gamma_m \frac{\partial n_{k_m}}{\partial a} \frac{\partial a^M}{\partial n} = \sum_{m=0}^{M} \gamma_m \Delta_{k_m} \frac{\partial a^M}{\partial n} = \Delta^M(a) \frac{\partial a^M}{\partial n} \tag{A.19}$$

That is:

$$\frac{\partial a^M}{\partial n} = \frac{1}{\Delta^M(a)} > 0 \tag{A.20}$$

and:

$$\frac{\partial^2 a^M}{\partial n^2} = -\frac{1}{\left(\Delta^M(a)\right)^2} \frac{\partial \Delta^M}{\partial a} \frac{\partial a^M}{\partial n} > 0 \tag{A.21}$$

We can conclude that $\delta^M(n)$ is non-increasing and concave. ■

We, finally, need to show that $\Delta^M(a)$ can be an arbitrarily close approximation to any $\Delta(a)$. Indeed, Δ^M consists of linear segments between the k_m and changes the slope at each k_m if the corresponding $\gamma_m > 0$. We construct two new functions, Δ^* and Δ^{**} with corresponding weights γ^* and γ^{**}. First, we divide the interval $[0, 1]$ into M equal intervals, beginning at 0, $1/M$, $2/M$, etc.; that is $k_m = m/M$. Each interval is hence described by $[m/M, (m + 1)/M]$ for $m = 0, 1, \ldots, M - 1$. We make two assumptions to simplify the analysis:

ASSUMPTION A.1 $\Delta(a)$ is differentiable on the entire interval $(0,1)$.[3]

ASSUMPTION A.2 The (left-side) derivative of $\Delta(a)$ is finite at $a = 1$.[4]

The slope (derivative) of Δ at point m/M is denoted by s_m for $m = 0, \ldots, M$ (in the case of s_0 the right-side derivative, and in the case of s_M the left-side derivative). We construct Δ^* by fixing the slopes of the different segments. Let the slope in interval $[0, 1/M]$ be s_0, in interval $[1/M, 2/M]$ s_1, and so on. Then, we can prove:

LEMMA A.3 Δ^* can be written as a weighted sum of the Δ_k:

$$\Delta^* = \sum_{m=0}^{M-1} \gamma_m^* \Delta_{k_m} \tag{A.22}$$

with each $\gamma_m^* \geq 0$. Further, $\Delta^*(a) \leq \Delta(a)$ for all $a \in [0,1]$.

PROOF The γ^* can be found explicitly, since for any given value of m only the Δ_k with $k \leq m/M$ will contribute to the slope:

$$s_m = \sum_{l=0}^{m} \gamma_l^* \frac{1}{1-k_l^2} \tag{A.23}$$

Solving for γ^*, we find:

$$\begin{aligned} & \gamma_0^* = s_0 \\ \gamma_0^* + \gamma_1^* \frac{1}{1-k_1^2} = s_1 \Rightarrow \quad & \gamma_1^* = (s_1 - s_0)\left(1 - k_1^2\right) \\ \gamma_0^* + \gamma_1^* \frac{1}{1-k_1^2} + \gamma_2^* \frac{1}{1-k_2^2} = s_2 \Rightarrow \quad & \gamma_2^* = (s_2 - s_1)\left(1 - k_2^2\right) \end{aligned} \tag{A.24}$$

By induction we see that $\gamma_m^* = (s_m - s_{m-1})\left(1 - k_m^2\right)$ for $1 \leq m \leq M-1$. That is, $\gamma_m^* \geq 0$.

Due to concavity of $\Delta(a)$, for any $a \in [m/M, (m+1)/M]$, $|\partial\Delta/\partial a| \geq s_m$ by construction. That is, the absolute value of the slope of $\Delta^*(a)$ is less than or equal to the the absolute value of the slope of $\Delta(a)$ for all $a \in [0,1]$. We know that $\Delta^*(1) = \Delta(1) = 0$. Hence, by continuity $\Delta^*(n) \leq \Delta(n)$ for all $a \in [0,1]$. ■

Δ^{**} is defined analogously to Δ^* with the slope of the segments fixed by the slope of Δ at the end of each interval: The slope in interval $[0, 1/M]$ is s_1, in interval $[1/M, 2/M]$ s_2, and so on. That leads us to:

[3] Without this assumption, we would have to operate with left-side and right-side derivatives at the points of non-differentiability. The essence of the analysis would not be affected, though.

[4] This assumption permits us to ignore Δ_1. Without the assumption γ_M^* and γ_M^{**} would have to be analyzed separately from the rest of the analysis, which would be onerous without adding anything substantial.

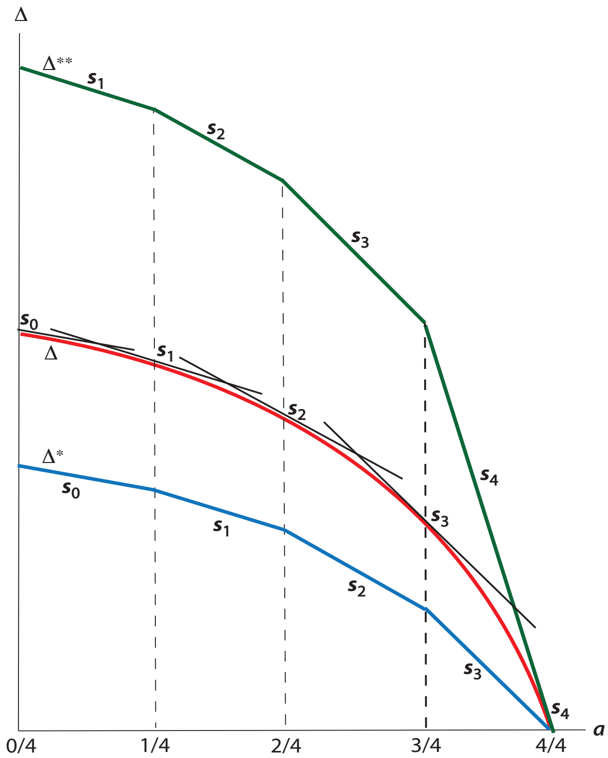

FIGURE A.1 Four-linear-intervals approximation

LEMMA A.4 Δ^{**} can be written as a weighted sum of the Δ_k:

$$\Delta^{**} = \sum_{m=0}^{M-1} \gamma_m^{**} \Delta_{k_m} \tag{A.25}$$

with each $\gamma_m^{**} \geq 0$. Further, $\Delta^{**}(a) \geq \Delta(a)$ for all $a \in [0,1]$

PROOF Analogous to the proof of lemma A.3. ■

The final step is to show that the two functions Δ^*, and Δ^{**} converge to Δ as M increases:

LEMMA A.5

$$\lim_{M\to\infty} \Delta^*(a) = \lim_{M\to\infty} \Delta^{**}(a) = \Delta(a), \quad a \in [0,1]. \tag{A.26}$$

PROOF The three functions Δ, Δ^*, and Δ^{**} are pictured in figure A.1 for $n = 4$. We want to calculate the size of the region between Δ^* and Δ^{**}. In the last interval, the area is:

$$A_4 = \frac{1}{4}\frac{s_4 - s_3}{4}\frac{1}{2} = \frac{1}{2}\frac{\Delta s_4}{4^2} \tag{A.27}$$

We have defined $\Delta s_m := s_m - s_{m-1}$. Similarly, in the third interval it is:

$$A_3 = \frac{\Delta s_4}{4^2} + \frac{1}{2}\frac{\Delta s_3}{4^2} \tag{A.28}$$

and for the second and first intervals:

$$A_2 = \frac{\Delta s_4}{4^2} + \frac{\Delta s_3}{4^2} + \frac{1}{2}\frac{\Delta s_2}{4^2} \tag{A.29}$$

and

$$A_1 = \frac{\Delta s_4}{4^2} + \frac{\Delta s_3}{4^2} + \frac{\Delta s_2}{4^2} + \frac{1}{2}\frac{\Delta s_1}{4^2} \tag{A.30}$$

The area A is then:

$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 \\ &= \frac{(2-1)\Delta s_2}{4^2} + \frac{(3-1)\Delta s_3}{4^2} + \frac{(4-1)\Delta s_4}{4^2} \\ &+ \frac{1}{2}\frac{\Delta s_1 + \Delta s_2 + \Delta s_3 + \Delta s_4}{4^2} \\ &= \frac{(2-1)\Delta s_2}{4^2} + \frac{(3-1)\Delta s_3}{4^2} + \frac{(4-1)\Delta s_4}{4^2} + \frac{1}{2}\frac{s_4 - s_0}{4^2} \end{aligned} \tag{A.31}$$

In general, with M intervals, the area is:

$$\begin{aligned} A &= \sum_{m=1}^{M} \frac{(m-1)\Delta s_m}{M^2} + \frac{1}{2}\frac{s_M - s_0}{M^2} \\ &\leq \frac{\Delta s^{max}}{M^2} \sum_{m=1}^{M} (m-1) + \frac{1}{2}\frac{s_M - s_0}{M^2} \\ &= \frac{\Delta s^{max}}{M^2}\frac{M(M-1)}{2} + \frac{1}{2}\frac{s_M - s_0}{M^2} \\ &= \frac{\Delta s^{max}}{2}\frac{M-1}{M} + \frac{1}{2}\frac{s_M - s_0}{M^2} \end{aligned} \tag{A.32}$$

Since Δ is differentiable, each Δs_m, and, hence, also the biggest one, Δs^{max}, approaches zero as n increases. In other words:

$$\lim_{M\to\infty} A = 0 \tag{A.33}$$

and the functions converge as M increases. ■

That is, Δ^M can approximate Δ to any desirable degree of precision. Hence, since $\Delta^M(a)$ implies a concave $\delta^M(n)$, by continuity a concave $\Delta(a)$ also implies concavity of the corresponding $\delta(n)$.