Testing the Tolerance Principle

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Tolerance Principle

Definition

Rule-based learning, such as past-tense acquisition, is commonly observed in language acquisition. But what leads to the use of the rules in the first place?

> **Tolerance Principle**: Let *R* be a rule that applies to N items, of which e are exceptions. *R* is productive if and only iff: $e \leq \theta_N$, where $\theta = \dfrac{N}{lnN}$ (Yang, 2016).

Examples:

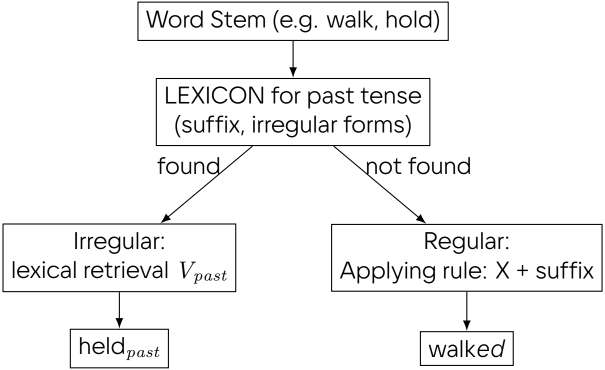
- If we have 20 verbs, and 5 are irregular verbs, will a rule be derived?
  - N = 20, e = 5, θ = $\dfrac{N}{lnN} = \dfrac{20}{ln(20)} \approx$ 6.7.
  - e $< \theta$, so a rule **will** be derived
- If we have 10 verbs, and 5 are irregular verbs, will a rule be derived?
  - N = 10, e = 5, θ = $\dfrac{N}{lnN} = \dfrac{10}{ln(10)} \approx$ 4.3.
  - e $> \theta$, so a rule **won't** be derived

Assumptions of the Tolerance Principle

- Why a Rule is deployed? A productive rule should be deployed when it delivers more efficient results than not using the rule.
    - For TP, more efficient = Faster
    - TP hypothesizes that a productive rule will reduce the average time of retrieving the target form.
- Dual-route model for the regular and irregular verb processing (Pinker & Prince, 1988).

Dual-route Model

- Regular Verbs: Processed by Rule Applying Mechanism
- Irregular Verbs: Processed through rote and associative memory
- Input goes into LEXICON for search, the LEXICON only contains the suffix and the irregular forms. If a match is found, then output the irregular form; else, apply the rule.



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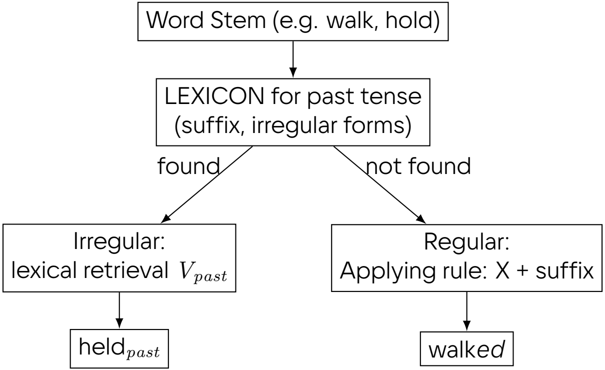
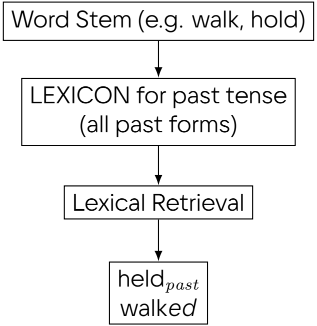
Rule VS No Rule

Rule-based Model



No-Rule Model



Intuition: Rule-based Model should save lexical retrieval time than No-Rule Model because there are fewer items to search in the LEXICON.

No Rule: Calculating the time complexity (T)

- **LEXICON structure:** All the lexical items are stored in a ranked list based on their frequency, with the most frequent items at the top.
- **Search function:** Serial Search process (Forster, 1976, 1992): to retrieve an item at position i, the model sequentially searches all the i-1 items ranked higher than i.
- **Intuition:** the less frequent words take longer to retrieve than the more frequent words.
- **Each word's average time complexity:** $p_i \cdot t_i$, p_i is the word's probability and t_i is its retrieval time.
- **Average time complexity for the lexicon list:** $T = \sum_{i=1}^{N} (p_i \cdot t_i)$.

No Rule: Calculating the time complexity (T)

- t_i: Assumes the rank hypothesis (Murray & Forster, 2004): the i-th ranked item takes i units of time to be retrieved, $t_i = r_i$
- p_i: Assumes the Zipfian distribution (Zipf, 1949): a word's frequency (f_i) times its rank (r_i) is a constant C: $C = f_i \cdot r_i$.
- Replacing f_i with $\frac{C}{r_i}$, $p_i = \frac{f_i}{\sum_{k=1}^{N} f_k} = \frac{\frac{C}{r_i}}{\sum_{k=1}^{N} \frac{C}{r_k}} = \frac{\frac{1}{r_i}}{\sum_{k=1}^{N} \frac{1}{r_k}}$.
- Insert: $T = \sum_{i=1}^{N} (p_i \cdot t_i) = \sum_{i=1}^{N} (\frac{\frac{1}{r_i}}{\sum_{k=1}^{N} \frac{1}{r_k}} \cdot r_i) = \sum_{i=1}^{N} (\frac{1}{\sum_{k=1}^{N} \frac{1}{r_k}})$
- $\frac{1}{\sum_{k=1}^{N} \frac{1}{r_k}}$ is Harmonic number H_N and Yang approximated $H_N \approx lnN$
- $T_{NoRule} \approx \frac{N}{lnN}$

Rule: Calculating the time complexity

- Rule-based model divides the time complexity into two parts: T_E for the exceptions and T_R for the rule-based items.
- Assuming there are N items and e exceptions ($e \leq N$).
- **Exceptions** are processed the same way in the no-rule model: $T_E \approx \frac{e}{lne} \cdot \frac{e}{N}$
- **Rule-based items** are assumed to have the same time complexity because they are reached after a thorough search of e exceptions: $T_R = e \cdot (1 - \frac{e}{N})$.
- $T_{Rule} = \frac{e}{lne} \cdot \frac{e}{N} + e \cdot (1 - \frac{e}{N})$

Deriving the Tolerance Principle

Assumption: more efficient = faster, so when $T_{Rule} \leq T_{NoRule}$ a rule will be deployed. Solving the inequation:

$$\frac{e}{N} \cdot \frac{e}{lne} + (1 - \frac{e}{N}) \cdot e \leq \frac{N}{lnN} \tag{1}$$

$$\frac{e}{N} \cdot (\frac{e}{lne} - e) + e \leq \frac{N}{lnN} \tag{2}$$

Since $\frac{e}{N} \cdot (\frac{e}{lne} - e) \leq 0$, therefore $e \leq \frac{N}{lnN}$.

When $e \leq \frac{N}{lnN}$, a rule will be deployed.

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Problems with estimation $\frac{N}{lnN}$

Mathematically: $\lim_{x \to \infty} (H_N - lnN) = \gamma$ where γ is Euler's constant $\approx$ 0.58

- The difference between H_N and lnN could be substantial for TP's calculation
- For example, when N = 10: $\frac{N}{lnN} \approx 4.34$, $\frac{N}{H_N} \approx 3.41$.
- When there are 4 exceptions:
  - lnN says **yes** can be a rule $(4 < 4.34)$.
  - H_N says **no** there can't be a rule $(4 > 3.41)$.

Problems with approximation of the inequation

Mathematically: $e \leq \frac{N}{lnN}$ is not the solution to $\frac{e}{N} \cdot (\frac{e}{lne} - e) + e \leq \frac{N}{lnN}$

- The difference between the **actual** solution of e and **estimated** $\frac{N}{lnN}$ could be substantial.
- For example, when N = 20: $\frac{N}{lnN} \approx 6.67$, therefore $e \leq 6.67$.
- However, the true solution to the inequation is $e \leq 8.73$.
- When there are 7 exceptions, can a rule be derived?

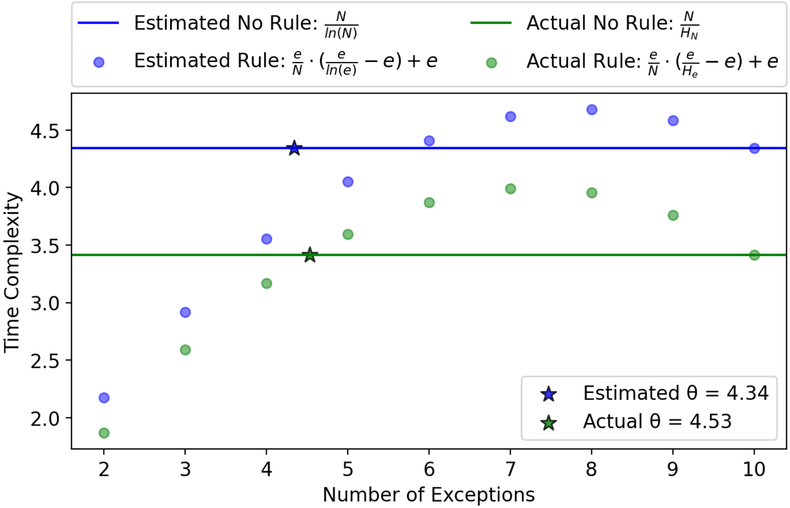
Testing it with different Ns

- Calculating the **actual** threshold θ using the Harmonic number by solving inequation:
$\frac{e}{N} \cdot \frac{e}{lne} + (1 - \frac{e}{N}) \cdot e \leq \frac{N}{lnN}$
- Comparing the result with $\frac{N}{lnN}$
- N = 10, 100, 1000

Table: The predicted θ and actual θ with different N

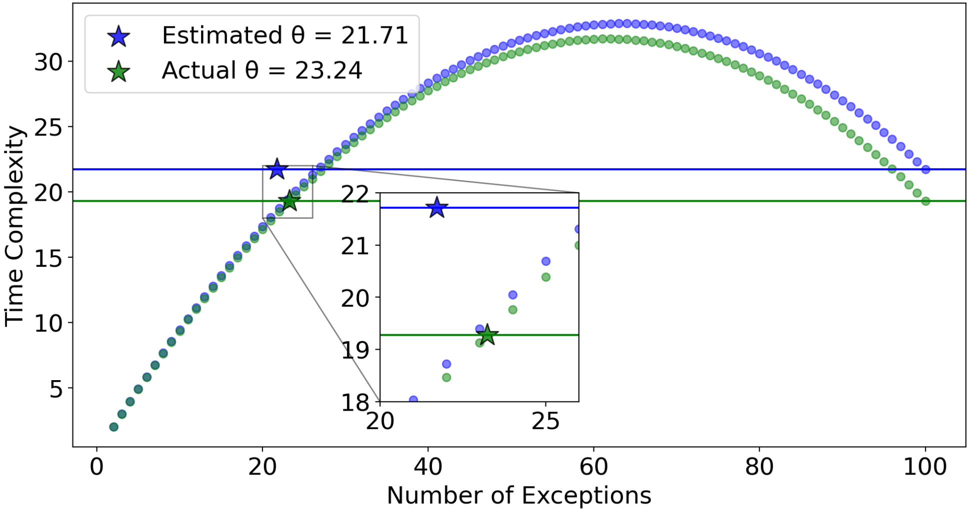
N	Predicted θ $N/ln(N)$	Actual θ
10	4.34	4.53
100	21.71	23.24
1,000	144.76	152.77

Plot: N = 10



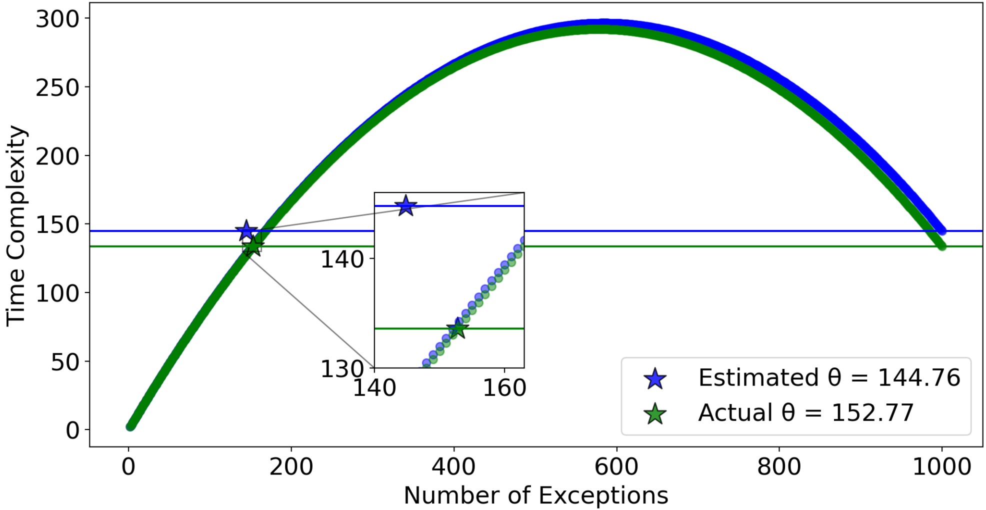
Estimated No Rule: $\frac{N}{ln(N)}$
Actual No Rule: $\frac{N}{H_N}$
Estimated Rule: $\frac{e}{N} \cdot (\frac{e}{ln(e)} - e) + e$
Actual Rule: $\frac{e}{N} \cdot (\frac{e}{H_e} - e) + e$
Time Complexity
4.5
4.0
3.5
3.0
2.5
2.0
2
3
4
5
6
7
8
9
10
Number of Exceptions
Estimated θ = 4.34
Actual θ = 4.53

Plot: N = 100



Estimated θ = 21.71
Actual θ = 23.24
Time Complexity
Number of Exceptions
0
20
40
60
80
100
5
10
15
20
25
30
22
21
20
19
18
20
25

Plot: N = 1000



300
250
200
150
100
50
0
Time Complexity
0
200
400
600
800
1000
Number of Exceptions
140
130
140
160
Estimated θ = 144.76
Actual θ = 152.77

Rank Matters

- **Intuition:** In TP's calculation, the t_i is the rank of the item. What if the exceptions all have high ranks? How would that affect the solution of the inequation?
- Creating a hypothetical list of 10 items whose distribution follows a Zipfian distribution: 1st ranked item has a frequency of 100 and the 10th ranked item has a frequency of 1.
- Calculate the T_{Rule} and T_{NoRule} using the formula and find the solution to the inequation $T_{Rule} \leq T_{NoRule}$.

Base Scenario: 10 items, All Exceptions

Intuition: No Rule should be derived

Item	Frequency	rank	Time Complexity $T = \sum_{i=1}^{N}(p_i \cdot t_i)$
Excep.	100	1	0.34 = 100/293 x 1
Excep.	50	2	0.34 = 50/293 x 2
Excep.	33	3	0.34 = 33/293 x 3
Excep.	25	4	0.34 = 25/293 x 4
Excep.	20	5	0.34 = 20/293 x 5
Excep.	17	6	0.35 = 17/293 x 6
Excep.	14	7	0.33 = 14/293 x 7
Excep.	13	8	0.35 = 13/293 x 8
Excep.	11	9	0.34 = 11/293 x 9
Excep.	10	10	0.34 = 10/293 x 10
Total	293		3.42

Scenario 1: 10 items, 7 exceptions

Intuition: No Rule should be derived.

No Rule

Verb	Frequency	rank	Time Complexity
Excep.	100	1	0.34
Excep.	50	2	0.34
Excep.	33	3	0.34
Excep.	25	4	0.34
Excep.	20	5	0.34
Excep.	17	6	0.35
Excep.	14	7	0.33
Regular	13	8	0.35
Regular	11	9	0.34
Regular	10	10	0.34
Total	293	T_{NoRule}	3.42

With a Rule

Excep.	Freq.	rank	Time Complexity
Excep.	100	1	0.24 = 100/293 x 1 x 0.7
Excep.	50	2	0.24 = 50/293 x 2 x 0.7
Excep.	33	3	0.24 = 33/293 x 3 x 0.7
Excep.	25	4	0.24 = 25/293 x 4 x 0.7
Excep.	20	5	0.24 = 20/293 x 5 x 0.7
Excep.	17	6	0.24 = 17/293 x 6 x 0.7
Excep.	14	7	0.23 = 14/293 x 7 x 0.7
Total			**1.67**
Regular			Time
Regular	13		
Regular	11		
Regular	10		
Total			**2.1** = 7 x 0.3
T_{Rule}			**3.77** = 1.67 + 2.1 $>$ 3.42

$T_{Rule} = 3.77$, $T_{NoRule} = 3.42$, since $T_{Rule} > T_{NoRule}$, **No rule will be derived.**

Scenario 2: 10 items, 7 exceptions

Intuition: No Rule should be derived.

No Rule

Item	Frequency	rank	Time Complexity
Excep.	100	1	0.34
Regular	50	2	0.34
Excep.	33	3	0.34
Excep.	25	4	0.34
Regular	20	5	0.34
Regular	17	6	0.35
Excep.	14	7	0.33
Excep.	13	8	0.35
Excep.	11	9	0.34
Excep.	10	10	0.34
Total	293	T_{NoRule}	3.42

With a Rule

Excep.	Freq.	rank	Time Complexity $T_E = \sum_{i=1}^{N}(p_i \cdot t_i) \cdot \frac{e}{N}$
Excep	100	1	0.24 = 100/293 x 1 x 0.7
Excep	33	2	0.16 = 33/293 x 2 x 0.7
Excep	25	3	0.18 = 25/293 x 3 x 0.7
Excep	14	4	0.13 = 14/293 x 4 x 0.7
Excep	13	5	0.16 = 13/293 x 5 x 0.7
Excep	11	6	0.16 = 11/293 x 6 x 0.7
Excep	10	7	0.17 = 10/293 x 7 x 0.7
Total			**1.19**

Regular			Time Complexity
Regular	50		$T_R = e \cdot (1 - \frac{e}{N})$
Regular	20		
Regular	17		
Total			**2.1** = 7 x 0.3
T_{Rule}			**3.29** = 1.19 + 2.1 $<$ 3.42

$T_{Rule} = 3.29$, $T_{NoRule} = 3.42$, since $T_{Rule} < T_{NoRule}$, **rule will be derived.**

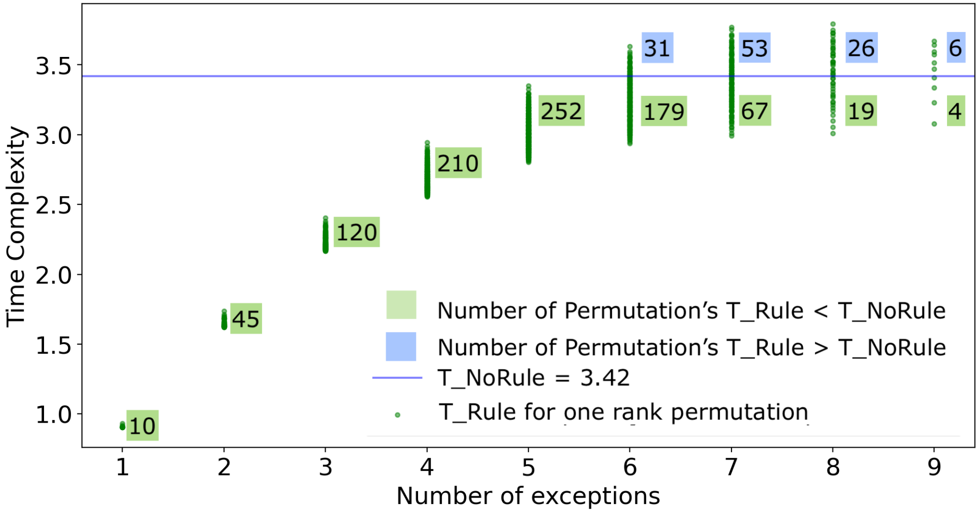
Discrepancies of the TP

- According to the TP, the number of exceptions (e) is the **only** factor determining whether a rule will be derived.
- Depending on the rank of the exceptions, the same number of exceptions would produce contradicting results. (e.g. In Scenario 1, $T_{Rule} < T_{NoRule}$, a rule will be derived. In Scenario 2, $T_{Rule} > T_{NoRule}$, a rule won't be derived.)
- Time complexity is not a fixed value. It varies depending on the rank permutation.

Testing the Rank Permutation

- **Observation:** When the regulars are of highest ranks, T_{Rule} reaches its **maximum**. When the regulars are of the lowest ranks, T_{Rule} reaches its **minimum**.
- Using N = 10 exhaustively calculate the T_{Rule} for all rank permutations with different numbers of exceptions
- Solve the inequation $T_{Rule} < T_{NoRule}$ to find the threshold θ.

Permuation: N = 10

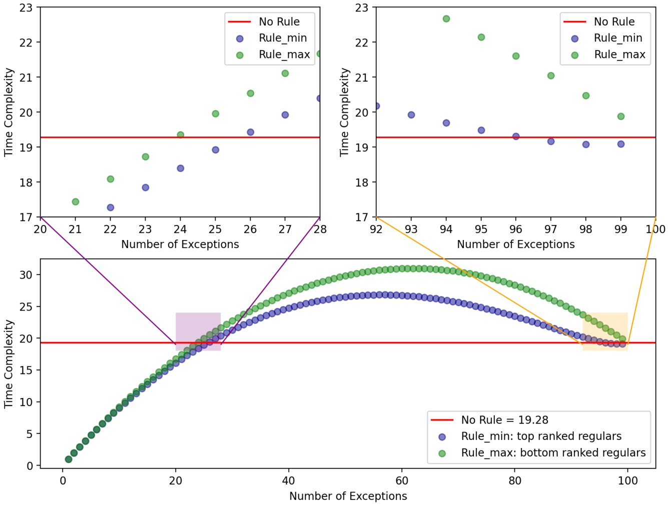


Further Test: N = 100, 1000

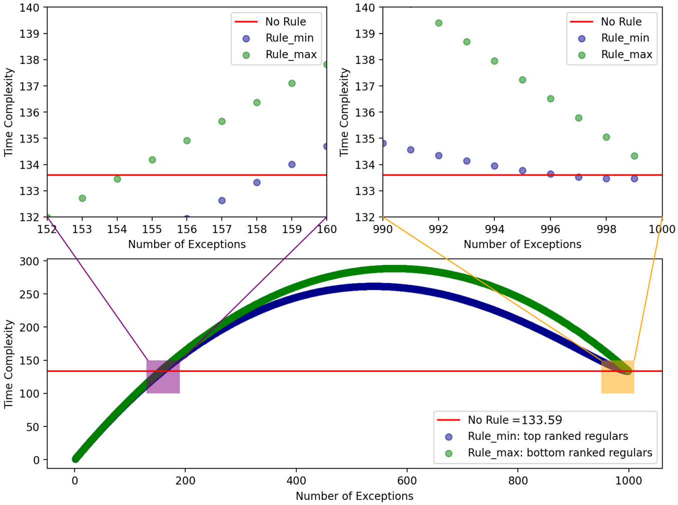
- Using N = 100, 1000 to calculate the $T_{Rule(MAX)}$ and $T_{Rule(MIN)}$ and find θ_{min} and θ_{max}.

	N = 100	N = 1000
T_{NoRule}	19.28	133.59
$\theta = N/ln(N)$	21.71	144.76
$T_{Rule(MAX)}$ (with integer θ)		
θ_{min}	23	154
θ_{max}	NA(> 100)	NA(> 1000)
$T_{Rule(MIN)}$ (with integer θ)		
θ_{min}	25	158
θ_{max}	97	997
A rule is derived when $e \leq \theta_{min}$ or $e \geq \theta_{max}$		

Permuation: N = 100



Permuation: N = 1000



Summary

Mathematical Discrepancies

- $\frac{N}{lnN}$ is not a proper estimation of the maximum number of exceptions.
- $e \leq \frac{N}{lnN}$ is not a proper solution to the inequation $T_{Rule} < T_{NoRule}$

When a rule is derived cannot be solely predicted on e

- Datasets with the same number of exceptions but different rank permutations can lead to contradicting results.

More Problems

Quadratic function of T_{Rule}

- TP's assumption assumes that the exceptions and the T_{Rule} has a linear relationship: if e is smaller than a threshold, $T_{Rule} < T_{NoRule}$, thus a rule will be derived.
- However, T_{Rule} is obviously quadratic, there are two sets of data that fit the rule-deriving criterion $T_{Rule} < T_{NoRule}$: $e \leq \theta_{min}$ or $e \geq \theta_{max}$.
- For example, when N = 100 and 1000, and $e > 97$ or $e > 997$, $T_{Rule} < T_{NoRule}$, a rule can be derived, which is impossible.
- The basic assumption of the TP is flawed.

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Motivation

One may argue that in real life many of the frequency permutations are not plausible and the quadratic pattern doesn't apply since there is a fixed number of exceptions.

Test the TP on past tense overregularization using children's corpus data.

Yang (2016)'s testing

- Yang (2016) applied the TP to explain past tense acquisition on Adam's and Eve's data (Brown, 1973; MacWhinney, 2000).
- The first overregularization error (e.g. **holded*) is seen as the sign or rule being deployed.
- Data: first recording to the recording of overregularization error
  - Adam: 2;3 - 2;11
  - Eve: 1:6 - 1:10*
- N: all the verb forms (including -*ing*, verb root, etc)the child produced.
- e: all the irregular verb forms the child produced.
- Results
  - Adam: N = 300, e = 57, $\theta = N/lnN \approx 53$, 57 $>$ 53, **failed** ✕
  - Eve: N = 163, e = 49, $\theta \approx 32$, **failed** ✕
- Explanation: Sampling errors

New Testing: Data

8 children's data from CHILDES.

	Age range	files	Total Verb Types (N)	Irregular Types (e)	Total Verb tokens	Irregular tokens
Adam	2;3 - 2;11	18	306	62	6,747	3,632
Eve[1]	1;6 - 1;8	5	93	36	564	337
Sarah	2;3 - 2;10	33	189	48	1,759	1,035
Peter	1;3 - 2;6	14	424	67	7,532	3,647
Naomi	1;3 - 1;11	20	128	43	1,240	757
Allison	1;5 - 2;11	6	88	36	612	335
April	1;10 - 2;1	2	50	19	128	62
Fraser	2;0 - 2;5	90	371	78	13,924	9,903

New Testing: Method

- Replicated Yang's method to count N and e.
- Compare e to N/lnN
- Compare e to $N/H(N)$
- Calculate T_{Rule} and T_{NoRule} using the verbs' actual rank and frequency and compare

New Testing: Results

	N	e	θ_p	$e < \theta_p$	θ_a	$e < \theta_a$	T_{NoRule}	T_{Rule}	$T_{Rule} < T_{NoRule}$
Adam	306	62	53.5	×	48.6	×	33.80	51.33	×
Eve	93	36	20.5	×	18.2	×	17.51	25.11	×
Sarah	189	48	36.1	×	32.5	×	25.65	37.81	×
Peter	424	67	70.1	✓	64	×	43.82	57.74	×
Naomi	128	43	26.4	×	23.6	×	19.63	31.23	×
Allison	88	36	19.7	×	17.4	×	18.24	34.72	×
April	50	19	12.8	×	11.1	×	14.64	14.29	✓
Fraser	371	78	62.7	×	57.1	×	26.34	60.03	×

$\theta_p = N/lnN$ is the TP predicted θ. $\theta_a = N/H_N$ is the actual θ.

Only Peter's actual $e < N/lnN$. Only April's $T_{Rule} < T_{NoRule}$

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Conclusion

On Hypothetical Data

- TP has several mathematical discrepancies that would lead to implausible results (e.g. when there are more than 97 exceptions in 100 items, a rule would be derived) or contradictory results (e.g. when there are 3 regulars in 10 items, if they rank 8,9,10, a rule can't be derived; otherwise a rule would be derived).

On Children's corpora Data

- Majority of the children's data don't conform to the TP's predictions.

Why won't TP work?

- Theoretical Assumption: a rule is derived to reduce time complexity.
  - Alternative 1: the rule is derived to reduce both time complexity and memory space.
  - Alternative 2: the rule is derived not for any utilitarian reasons.

- Operational Assumption: the time complexity has a linear relationship with the number of exceptions. The calculation relies on **dual-route model**, **serial search process**, **rank hypothesis** and **Zipfian distribution**.
  - Dual-route model vs Connectionist model
  - Serial search vs Parallel process
  - Modify t_i instead of using rank
  - Zipfian distribution doesn't really apply to small datasets

References

Brown, R. (1973). *A first language: the early stages*. Cambridge, MA: Harvard University Press.

Forster, K. I. (1976). Accessing the mental lexicon. *New approaches to language mechanisms*, 257–287.

Forster, K. I. (1992). Memory-addressing mechanisms and lexical access. *Orthography, phonology, morphology, and meaning*, 413.

MacWhinney, B. (2000). *The* CHILDES P*roject: Tools for analyzing talk*. *transcription format and programs* (Vol. 1). Psychology Press.

Murray, W. S., & Forster, K. I. (2004). Serial mechanisms in lexical access: the rank hypothesis. *Psychological Review*, *111*(3), 721.

Pinker, S., & Prince, A. (1988). On language and connectionism: Analysis of a parallel distributed processing model of language acquisition. *Cognition*, *28*(1-2), 73–193.

Yang, C. (2016). *The price of linguistic productivity: How children learn to break the rules of language*. MIT Press.

Zipf, G. K. (1949). *Human behavior and the principle of least effort*. Cambridge: Addison-Wesley Press.

Questions?

Thanks!